THE ANALYST.

VOL. X. JULY, 1883. No. 4

A METHOD OF DEMONSTRATING CERTAIN PROPERTIES OF POLYNOMIALS.

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IN former articles of mine (see ANALYST, Vol. VII, p. 39 and Vol. IX, p. 141), it has been shown that when the coefficients of a polynomial are regarded as parallel forces acting positively or negatively in any coordinate direction at points whose rectangular coordinates are proportional to the exponents of the variables, the lever arm of the system of forces with respect to the coordinate planes, its "radius of gyration" or "quadratic radius" with respect to coordinate planes passed through the centre of forces of the system, and its "cubic radius" also, are such that if we multiply two or more polynomials together, the arm and the radii for the product can be found from those of the factors in a very simple manner. This has received an important application to the subject of probability. The coefficient or force at each point is regarded as the probability that an error which occurs will fall at that point. Since probabilities are always positive, and parallel forces acting in one direction may be represented by the force of gravity, and gravity is proportional to mass, the coefficients of the polynomial may be conc'd of as the masses of a system of material points. The centre of forces then becomes the centre of gravity of the system. It is the point whose coordinate in any direction is the *arithmetical mean* of the corresponding coordinates of all the points of error, each taken with a weight proportional to the probability of its occurrence. The sum of the probabilities of all possible errors is unity, that is, certainty. We shall assume that whether the coefficients of a polynomial represent probabilities or not, their algebraic sum is always unity. This does not impair the generality in any case, every polynomial being reducible to this form by dividing it through by the algebraic

sum of its coefficients. The methods of demonstrating the properties we have referred to have hitherto been special and restricted, but we shall now show that they can all be replaced by one which is both simple and comprehensive, giving many new properties which could not well be otherwise proved.

Let the polynomials have three variables r, s, t, with exponents a, b, c proportional to the coordinates x, y, z, locating the points anywhere in space of three dimensions. The properties for one and two dimensions are special cases under this, and will not need separate proof. Using a notation similar to that of ANALYST, IX. 34, we can write the first polynomial factor

$$u = \Sigma_{c=-m}^{c=m} \Sigma_{b=-m}^{b=m} \Sigma_{a=-m}^{a=m} (L_{a,\,b,\,c}\, r^a s^b t^c), \tag{1}$$

the coefficient L of each term being distinguished by sub-indices equal to the exponents of the variables in that term. The units of measure in the x, y, z directions are Δx, Δy, Δz, which may be taken of any convenient magnitude or even infinitesimals dx, dy, dz, and are not necessarily equal to each other; and a, b, c are numbers, positive or negative, integral or fractional, such that for any point

$$x = a\Delta x, \qquad y = b\Delta y, \qquad z = c\Delta z. \tag{2}$$

The number m is taken so large that a, b or c will not exceed it. The polynomial (1) can be understood to include all the points in a rectangular block. When any points are not to be included, we only have to suppose that for each of them the coefficient L is zero. We may take m large enough to include either of two polynomial factors u and v. Denoting the coefficients in v by L', we write

$$v = \Sigma_{c=-m}^{c=m} \Sigma_{b=-m}^{b=m} \Sigma_{a=-m}^{a=m} (L'_{a,\,b,\,c}\, r^a s^b t^c).$$

Let l denote the coefficients in the product of u and v, then

$$uv = \Sigma_{c=-2m}^{c=2m} \Sigma_{b=-2m}^{b=2m} \Sigma_{a=-2m}^{a=2m} (l_{a,\,b,\,c}\, r^a s^b t^c).$$

Since the limits of summation in either factor will remain the same throughout our present investigation, and those in the product likewise, we will omit writing them and merely suppose that the summation extends to all the terms in the polynomial. Also, to save repetition, we omit to write the sub-indices a, b, c after L, L' and l, and we denote $\Sigma\Sigma\Sigma$ by Σ_3 and $\Sigma\Sigma$ by Σ_2. The two polynomials now are

$$u = \Sigma_3(Lr^a s^b t^c), \qquad v = \Sigma_3(L'r^a s^b t^c), \tag{3}$$

and their product is

$$uv = \Sigma_3(lr^a s^b t^c). \tag{4}$$

We shall not regard the coefficients exclusively as forces or masses, but as quantities which are located at points in space, and which are to be multiplied into certain powers or products of their coordinates. Adding together

all the products thus obtained in any one polynomial, we call their sum a *moment* of the system of coefficients. For instance, denoting the coordinates as in (2),

$$\Sigma_3(Lx) = \Sigma_3(aL)\Delta x$$

is the x moment of the coefficients L; that is, the sum of the products formed by multiplying each L into its abscissa x. Likewise

$$\Sigma_3(Lx^2y) = \Sigma_3(a^2bL)(\Delta x)^2\Delta y$$

is the x^2y moment; and so on.

The variables are independent of the coefficients, and of each other. Differentiating (3) and (4) with respect to r and multiplying the results by r, we get

$$\left.\begin{array}{c} r\left(\dfrac{du}{dr}\right) = \Sigma_3(aLr^as^bt^c), \quad r\left(\dfrac{dv}{dr}\right) = \Sigma_3(aL'r^as^bt^c), \\ r\left\{ v\left(\dfrac{du}{dr}\right)+u\left(\dfrac{dv}{dr}\right) \right\} = \Sigma_3(alr^as^bt^c). \end{array}\right\} \tag{5}$$

As these relations are true for any values assigned to the variables, we may assume

$$r = 1, \quad s = 1, \quad t = 1, \quad \therefore u = 1, \quad v = 1, \tag{6}$$

since by hypothesis $\Sigma_3L = 1$, $\Sigma_3L' = 1$. Then, denoting by $(du \div dr)_1$ &c. what $du \div dr$ &c. become when $r = s = t = 1$, we get from (5)

$$\left.\begin{array}{c} \left(\dfrac{du}{dr}\right)_1 = \Sigma_3(aL), \quad \left(\dfrac{dv}{dr}\right)_1 = \Sigma_3(aL'), \\ \left(\dfrac{du}{dr}\right)_1 + \left(\dfrac{dv}{dr}\right)_1 = \Sigma_3(al). \end{array}\right\} \tag{7}$$

The first member of the last equation being the sum of those of the two first, we have

$$\Sigma_3(al)\Delta x = \Sigma_3(aL)\Delta x + \Sigma_3(aL')\Delta x, \tag{8}$$

that is, giving $a\Delta x$ its value from (2), the x moment in the product is equal to the sum of the x moments in the two factors. This is the theorem respecting the lever arms. (ANALYST, Vol. VII, p. 80.) When the sum of the masses in a system is unity, the statical moment of the system is numerically the same as its lever arm. By differentiating (3) and (4) with respect to s and t, we could get relations like (8) for the y and z moments. But owing to the symmetry of (3) with respect to a, b and c, what is proved for one coordinate direction is proved, *mutatis mutandis*, for all. Hence we have

$$\left.\begin{array}{c} \Sigma_3(lx) = \Sigma_3(Lx)+\Sigma_3(L'x), \quad \Sigma_3(ly) = \Sigma_3(Ly)+\Sigma_3(L'y), \\ \Sigma_3(lz) = \Sigma_3(Lz)+\Sigma_3(L'z). \end{array}\right\} \tag{9}$$

With polynomials of only two variables the points for each are all in one XY plane. Omitting t and c from (3) and (4), we get as above

$$\Sigma_2(lx) = \Sigma_2(Lx)+\Sigma_2(L'x), \quad \Sigma_2(ly) = \Sigma_2(Ly)+\Sigma_2(L'y). \tag{10}$$

(Compare Vol. VII, p. 45.) And with only one variable, the points being ranged along a straight line or x axis (Vol. VII, p. 22),

$$\Sigma(lx) = \Sigma(Lx) + \Sigma(L'x). \tag{11}$$

We have hitherto supposed that the origins of coordinates, or places of $L_{0,0,0}$ and $L'_{0,0,0}$ in (1) and (3), are taken anywhere at pleasure, but a great advantage will be gained by taking them so that the sum of the products Lx on one side of the origin is equal to the sum of those on the other side; and likewise for the $L'x$, the Ly, &c. This reduces the lever arms to zero, and locates the origins at the centres of forces of the two systems, which are the centres of gravity when the coefficients L and L' are positive and regarded as masses. It gives

$$\left.\begin{array}{lll} \Sigma_3(Lx) = 0, & \Sigma_3(Ly) = 0, & \Sigma_3(Lz) = 0, \\ \Sigma_3(L'x) = 0, & \Sigma_3(L'y) = 0, & \Sigma_3(L'z) = 0. \end{array}\right\} \tag{12}$$

The points thus determined will henceforth be taken as origins. Then by (2), (7) and (9) we have in the two factors

$$\left.\begin{array}{lll} \left(\dfrac{du}{dr}\right)_1 = 0, & \left(\dfrac{du}{ds}\right)_1 = 0, & \left(\dfrac{du}{dt}\right)_1 = 0, \\ \left(\dfrac{dv}{dr}\right)_1 = 0, & \left(\dfrac{dv}{ds}\right)_1 = 0, & \left(\dfrac{dv}{dt}\right)_1 = 0, \end{array}\right\} \tag{13}$$

and consequently in the product

$$\Sigma_3(lx) = 0, \quad \Sigma_3(ly) = 0, \quad \Sigma_3(lz) = 0, \tag{14}$$

the origin of coordinates or place of $l_{0,0,0}$ in the product being thus the centre of forces of the whole system of coefficients l, in the same sense as the origins in the factors are centres of forces for L and L'.

Now differentiating (5) with respect to r and multiplying the results by r, we get

$$\left.\begin{array}{l} r\left(\dfrac{du}{dr}\right)+r^2\left(\dfrac{d^2u}{dr^2}\right) = \Sigma_3(a^2Lr^as^bt^c), \\ r\left(\dfrac{dv}{dr}\right)+r^2\left(\dfrac{d^2v}{dr^2}\right) = \Sigma_3(a^2L'r^as^bt^c), \\ r\left\{ v\left(\dfrac{du}{dr}\right)+u\left(\dfrac{dv}{dr}\right) \right\} +r^2\left\{ v\left(\dfrac{d^2u}{dr^2}\right)+2\left(\dfrac{du}{dr}\cdot\dfrac{dv}{dr}\right)+u\left(\dfrac{d^2v}{dr^2}\right) \right\} \\ \qquad = \Sigma_3(a^2lr^as^bt^c), \end{array}\right\} \tag{15}$$

and giving the quantities the values from (6) and (13),

$$\left.\begin{array}{l} \left(\dfrac{d^2u}{dr^2}\right)_1 = \Sigma_3(a^2L), \quad \left(\dfrac{d^2v}{dr^2}\right)_1 = \Sigma_3(a^2L'), \\ \left(\dfrac{d^2u}{dr^2}\right)_1+\left(\dfrac{d^2v}{dr^2}\right)_1 = \Sigma_3(a^2l). \end{array}\right\} \tag{16}$$

Consequently

$$\Sigma_3(a^2 l)(\Delta x)^2 = \Sigma_3(a^2 L)(\Delta x)^2 + \Sigma_3(a^2 L')(\Delta x)^2,$$

and applying (2), we have in the three coordinate directions,

$$\left.\begin{array}{l}\Sigma_3(lx^2) = \Sigma_3(Lx^2)+\Sigma_3(L'x^2),\ \Sigma_3(ly^2) = \Sigma_3(Ly^2)+\Sigma_3(L'y^2),\\ \qquad\Sigma_3(lz^2) = \Sigma_3(Lz^2)+\Sigma_3(L'z^2).\end{array}\right\}\quad(17)$$

The x^2 moment in the product of two polynomials is equal to the sum of the x^2 moments in the two factors, when the origins are at the centres of forces; and the like is true for the y^2 and z^2 moments. This is the theorem respecting the radii of gyration, or quadratic radii (AN., Vol. VII, p. 81). When the mass of a system is unity, its moment of inertia is numerically equal to the square of its radius of gyration. The theorem holds true also when the moments are taken not with respect to the coordinate planes, but with respect to the coordinate axes or to the origin. (Vols. VII, p. 82 and IX, p. 67.) If the points for each polynomial are all in one plane, (17) reduces to

$$\Sigma_2(lx^2) = \Sigma_2(Lx^2)+\Sigma_2(L'x^2),\quad \Sigma_2(ly^2) = \Sigma_2(Ly^2)+\Sigma_2(L'y^2)\quad(18)$$

(Vol. VII, p. 78); and if all are in one straight line (VII, p. 22),

$$\Sigma(lx^2) = \Sigma(Lx^2)+\Sigma(L'x^2).\qquad(19)$$

This property of polynomials affords the simplest explanation of that "law of great numbers," so important in the theory of probability. It proves rigorously that with any given law of facility of error, the "quadratic mean error" of the arithmetical mean of n observations varies inversely as $\sqrt{n}$. (Vols. VIII, p. 4 and IX, p. 168.) Hence, by increasing the number of observations, we can diminish indefinitely the probable error of the mean.

Next, differentiating (15) with respect to r and then multiplying by r, we get

$$\left.\begin{array}{l} r\left(\dfrac{du}{dr}\right)+3r^2\left(\dfrac{d^2u}{dr^2}\right)+r^3\left(\dfrac{d^3u}{dr^3}\right) = \Sigma_3(a^3Lr^as^bt^c),\\ r\left(\dfrac{dv}{dr}\right)+3r^2\left(\dfrac{d^2v}{dr^2}\right)+r^3\left(\dfrac{d^3v}{dr^3}\right) = \Sigma_3(a^3L'r^as^bt^c),\\ r\left\{v\left(\dfrac{du}{dr}\right)+u\left(\dfrac{dv}{dr}\right)\right\}+3r^2\left\{v\left(\dfrac{d^2u}{dr^2}\right)+2\left(\dfrac{du}{dr}\cdot\dfrac{dv}{dr}\right)+u\left(\dfrac{d^2v}{dr^2}\right)\right\}\\ +r^3\left\{v\left(\dfrac{d^3u}{dr^3}\right)+3\left(\dfrac{dv}{dr}\cdot\dfrac{d^2u}{dr^2}+\dfrac{du}{dr}\cdot\dfrac{d^2v}{dr^2}\right)+u\left(\dfrac{d^3v}{dr^3}\right)\right\}\\ \qquad\qquad = \Sigma_3(a^3lr^as^bt^c).\end{array}\right\}\quad(20)$$

By means of (6) and (13) this is reduced to

$$3\left(\frac{d^2u}{dr^2}\right)_1+\left(\frac{d^3u}{dr^3}\right)_1 = \Sigma_3(a^3L),\ 3\left(\frac{d^2v}{dr^2}\right)_1+\left(\frac{d^3v}{dr^3}\right)_1 = \Sigma_3(a^3L'),$$

$$3\left\{\left(\frac{d^2u}{dr^2}\right)_1+\left(\frac{d^2v}{dr^2}\right)_1\right\}+\left(\frac{d^3u}{dr^3}\right)_1+\left(\frac{d^3v}{dr^3}\right)_1 = \Sigma_3(a^3l),$$

and consequently

$$\Sigma_3(a^3 l)(\Delta x)^3 = \Sigma_3(a^3 L)(\Delta x)^3 + \Sigma_3(a^3 L')(\Delta x)^3,$$

so that applying (2), we have

$$\left.\begin{array}{l} \Sigma_3(lx^3) = \Sigma_3(Lx^3) + \Sigma_3(L'x^3),\ \Sigma_3(ly^3) = \Sigma_3(Ly^3) + \Sigma_3(L'y_3), \\ \Sigma_3(lz^3) = \Sigma_3(Lz^3) + \Sigma_3(L'z^3). \end{array}\right\} \quad (21)$$

The x^3 moment in the product is equal to the sum of the x^3 moments in the two factors; and so too for the y^3 and the z^3 moments. In space of two dimensions (21) becomes

$$\Sigma_2(lx^3) = \Sigma_2(Lx^3) + \Sigma_2(L'x^3), \quad \Sigma_2(ly^3) = \Sigma_2(Ly^3) + \Sigma_2(L'y^3), \quad (22)$$

and in space of one dimension

$$\Sigma(lx^3) = \Sigma(Lx^3) + \Sigma(L'x^3). \quad (23)$$

This is the theorem respecting the "cubic radii". (ANALYST, IX, 161.)

Relations such as these can be proved for moments of the 4th and higher orders, only they will not be quite so simple. Differentiating (20) with respect to r, multiplying by r and applying (6) and (13), we get

$$7\left[\frac{d^2u}{dr^2}\right]_1 + 6\left[\frac{d^3u}{dr^3}\right]_1 + \left[\frac{d^4u}{dr^4}\right]_1 = \Sigma_3(a^4 L),$$

$$7\left[\frac{d^2v}{dr^2}\right]_1 + 6\left[\frac{d^3v}{dr^3}\right]_1 + \left[\frac{d^4v}{dr^4}\right]_1 = \Sigma_3(a^4 L'),$$

$$7\left\{\left[\frac{d^2u}{dr^2}\right]_1 + \left[\frac{d^2v}{dr^2}\right]_1\right\} + 6\left\{\left[\frac{d^3u}{dr^3}\right]_1 + \left[\frac{d^3v}{dr^3}\right]_1\right\} + \left[\frac{d^4u}{dr^4}\right]_1$$
$$+ \left[\frac{d^4v}{dr^4}\right]_1 + 6\left[\frac{d^2u}{dr^2}\right]_1\left[\frac{d^2v}{dr^2}\right]_1 = \Sigma_3(a^4 l),$$

and consequently by help of (16),

$$\Sigma_3(lx^4) = \Sigma_3(Lx^4) + \Sigma_3(L'x^4) + 6\Sigma_3(Lx^2)\Sigma_3(L'x^2). \quad (24)$$

The x^4 moment in the product is equal to the sum of the x^4 moments in the two factors, plus 6 times the product of their x^2 moments; and similarly for the y^4 and z^4 moments. In space of only one or two dimensions the relation is of corresponding form. In the same way it can be shown that

$$\Sigma_3(lx^5) = \Sigma_3(Lx^5) + \Sigma_3(L'x^5) + 10[\Sigma_3(Lx^2)\Sigma_3(L'x^3) + \Sigma_3(Lx^3)\Sigma_3(L'x^2)]. \quad \ldots\ldots (25)$$

In general, the moment of the nth order in the product will be expressed in terms of the moments of the nth and lower orders in the two factors. Hence, if some of the coefficients L and L' are negative, in such manner as to reduce to zero the moments of all orders up to and including the nth in both factors, the same moments in the product will also be zero. If they are zero in any one polynomial, they will be zero in all the powers of that polynomial. This is the property which I otherwise demonstrated in ANALYST, VI, 145 and VII, 105. The quantities there denoted by b_n, &c. are the same as $\Sigma(a^n L)$, &c., in our present notation.

Next let (5) be differentiated with respect to s. Multiplying the results by s we have

$$\left.\begin{aligned} rs\left[\frac{d^2u}{dr\,ds}\right] &= \Sigma_3(abLr^as^bt^c), \quad rs\left[\frac{d^2v}{dr\,ds}\right] = \Sigma_3(abL'r^as^bt^c), \\ rs\left\{v\left[\frac{d^2u}{dr\,ds}\right]+\frac{du}{dr}\cdot\frac{dv}{ds}+\frac{du}{ds}\cdot\frac{dv}{dr}+u\left[\frac{d^2v}{dr\,ds}\right]\right\} &= \Sigma_3(ablr^as^bt^c), \end{aligned}\right\} \quad (26)$$

and applying (6) and (13),

$$\left.\begin{aligned} \left[\frac{d^2u}{dr\,ds}\right]_1 &= \Sigma_3(abL), \quad \left[\frac{d^2v}{dr\,ds}\right]_1 = \Sigma_3(abL'), \\ \left[\frac{d^2u}{dr\,ds}\right]_1 &+ \left[\frac{d^2v}{dr\,ds}\right]_1 = \Sigma_3(abl), \end{aligned}\right\} \quad (27)$$

$$\therefore\ \Sigma_3(abl)\Delta x\Delta y = \Sigma_3(abL)\Delta x\Delta y + \Sigma_3(abL')\Delta x\Delta y,$$

and for the three coordinates taken two and two,

$$\left.\begin{aligned} \Sigma_3(lxy) = \Sigma_3(Lxy)+\Sigma_3(L'xy),\ \Sigma_3(lxz) &= \Sigma_3(Lxz)+\Sigma_3(L'xz), \\ \Sigma_3(lyz) = \Sigma_3(Lyz)+\Sigma_3(L'yz). \end{aligned}\right\} \quad (28)$$

The xy moment in the product is equal to the sum of the xy moments in the two factors; and so also for the xz and yz moments. For points in a plane, (28) reduces to the single relation

$$\Sigma_2(lxy) = \Sigma_2(Lxy)+\Sigma_2(L'xy). \quad (29)$$

Again, differentiating either (15) with respect to s or (26) with respect to r, multiplying the results by s in the one case or r in the other, and applying (6) and (13), we find

$$\left.\begin{aligned} \left[\frac{d^2u}{dr\,ds}\right]_1+\left[\frac{d^3u}{dr^2ds}\right]_1 &= \Sigma_3(a^2bL), \quad \left[\frac{d^2v}{dr\,ds}\right]_1+\left[\frac{d^3v}{dr^2ds}\right]_1 \\ &= \Sigma_3(a^2bL'), \\ \left[\frac{d^2u}{dr\,ds}\right]_1+\left[\frac{d^2v}{dr\,ds}\right]_1+\left[\frac{d^3u}{dr^2ds}\right]_1+\left[\frac{d^3v}{dr^2ds}\right]_1 &= \Sigma_3(a^2bl), \end{aligned}\right\} \quad (30)$$

$$\therefore\ \Sigma_3(lx^2y) = \Sigma_3(Lx^2y)+\Sigma_3(L'x^2y). \quad (31)$$

The x^2y moment in the product is equal to the sum of the x^2y moments in the two factors. The same is true of the x^2z, xy^2, y^2z, xz^2 and yz^2 moments. For points in a plane these six are reduced to two,

$$\Sigma_2(lx^2y)=\Sigma_2(Lx^2y)+\Sigma_2(L'x^2y),\ \Sigma_2(lxy^2)=\Sigma_2(Lxy^2)+\Sigma_2(L'xy^2). \quad (32)$$

Let us now differentiate (26) with respect to t, multiplying the results by t, and apply (6) and (13); this gives

$$\left.\begin{aligned} \left\{\frac{d^3u}{drdsdt}\right\}_1 &= \Sigma_3(abcL), \quad \left\{\frac{d^3v}{drdsdt}\right\}_1 = \Sigma_3(abcL'), \\ \left\{\frac{d^3u}{drdsdt}\right\}_1 &+ \left\{\frac{d^3v}{drdsdt}\right\}_1 = \Sigma_3(abcl), \end{aligned}\right\} \quad (33)$$

$$\therefore\ \Sigma_3(lxyz) = \Sigma_3(Lxyz) + \Sigma_3(L'xyz). \quad (34)$$

The xyz moment in the product is equal to the sum of the xyz moments in the two factors.

It is evident that such properties as (17), (21), (28), (31) and (34) can be readily extended to the product of any number of polynomial factors, the moment in the final product being equal to the sum of those in all the factors. If a polynomial is raised to the nth power, the moment for that power is n times as great as for the first power.

We have hitherto supposed that the rectangular coordinate axes are taken in any convenient directions, provided only that the origin must be at the centre of forces, which is the centre of gravity when the coefficients L and L' are all positive and regarded as the masses of material points. But in finding the law of probability of errors in space of two or three dimensions, the equations are simplified when we assume the axes to coincide with the "free axes," or principal axes through the centre of gravity. (ANALYST, VIII, 43 and IX, 38.) All the moments in (28) or (29) are thereby reduced to zero. Then by (27)

$$\left.\begin{array}{lll} \left\{\dfrac{d^2u}{dr\,ds}\right\}_1 = 0, & \left\{\dfrac{d^2u}{dr\,dt}\right\}_1 = 0, & \left\{\dfrac{d^2u}{ds\,dt}\right\}_1 = 0, \\ \left\{\dfrac{d^2v}{dr\,ds}\right\}_1 = 0, & \left\{\dfrac{d^2v}{dr\,dt}\right\}_1 = 0, & \left\{\dfrac{d^2v}{ds\,dt}\right\}_1 = 0. \end{array}\right\} \tag{35}$$

The positions of the free axes can be found for any given polynomial by known methods applicable to bodies of three dimensions, for which see chapter II of Vol. II of Poisson's *Traité de Mécanique*. For points in a plane the formula will be, as in the ANALYST above cited,

$$\tan 2\varphi = \frac{2\Sigma_2(Lxy)}{\Sigma_2(Lx^2)-\Sigma_2(Ly^2)}, \tag{36}$$

where φ is the angle which a free axis makes with the assumed X axis, and has two values differing by 90°. The moment in the numerator of (36) is of the same form as those in (29), while the moments in the denominator are like those in (18). When (36) is applied to the nth power of a polynomial, both numerator and denominator will be, as we have shown, n times as great as they were for the first power, so that the value of φ will be unchanged; that is to say, the free axes of the material points l in the expans'n will make with the assumed axes, the same angles which those of the points L did in the given polynomial. This is an improved demonstration of the property which I inferred from somewhat different considerations in ANALYST, Vol. VIII, p. 48. A similar property holds true for powers of a polynomial of three variables, occupying points in space of three dimensions. (Vol. IX, p. 67.) According to the formulas given by Poisson, it is manifest that the position of the free axes is not changed when the moments

used as data, of the same form as those in (17) and (28), are all multiplied by a constant number.

The positions of the free axes in the final product of any number of different polynomial factors will be given by the same formulas as above, when for the moments used we substitute the sums obtained by adding together the corresponding moments for all the factors.

When the free axes are taken as coordinate axes, simple properties like those we have already found can be demonstrated for moments of still higher orders. By the same process of differentiation and multiplication, and then applying (6), (13) and (35), it is found that

$$\Sigma_3(lx^3y) = \Sigma_3(Lx^3y)+\Sigma_3(L'x^3y), \tag{37}$$

the like being true of course for the x^3z, xy^3, y^3z, xz^3 and yz^3 moments. It is also found that

$$\Sigma_3(lx^2yz) = \Sigma_3(Lx^2yz)+\Sigma_3(L'x^2yz), \tag{38}$$

the same being true of the xy^2z and xyz^2 moments.

For the form x^2y^2 the relation is more complex, namely

$$\Sigma_3(lx^2y^2) = \Sigma_3(Lx^2y^2)+\Sigma_3(L'x^2y^2)+\Sigma_3(Lx^2)\Sigma_3(L'y^2) + \Sigma_3(Ly^2)\Sigma_3(L'x^2). \tag{39}$$

The like holds true for the x^2z^2 and y^2z^2 moments.

CORRESPONDENCE.

Editor Analyst:

If not trespassing too much upon your space permit me to reply in a few words to the letters of Mr. Adcock and Prof. Judson, in the last number of the ANALYST.

Mr. Adcock, after quoting from my original statement of the paradox, "now if $a = \infty$, $u = 0$ independently of x", adds "This I deny." His first answer begins, "When $u = 0$, independently of x it is not a function of x", from which it seems that he was at that time content to accept the equation $u = 0$. It is difficult to see what he wishes now to substitute for this eq'n, for he goes on to say, "In this case, u = actual zero, or an infinitesimal." Perhaps he means only to deny t e clause "independently of x"; for he remarks "the rate at which these infinitesimals change their value is $du \div dx = \cos ax$." But this is a mere restatement of the paradox; viz., that u has a finite rate of change, and yet no finite change in value.

Mr. Adcock's "private interpretation" of the form $\cos \infty$, as "indetermi-